Class 22, given on Feb 22, 2010, for Math 13, Winter 2010

1. Surface integrals of vector fields

Recall that we motivated the concept of a surface integral of a vector field by discussing the flux of a vector field across a surface. If the vector field is thought of as the amount of fluid which is flowing per unit time, the flux represents the total amount of fluid passing through the surface per unit time. In the course of discussing flux, we saw that a choice of normal vector to a surface influences the sign of the flux. This led us to discuss orientation, which is a choice of side of S, with the precise definition being a continuous choice of unit normal vector $\mathbf{n}$ of S, and then we examined how we could actually compute normal vectors from a parameterization of a surface.

Of course, a surface integral of a vector field $\mathbf{F}$ across a surface S in $\mathbb{R}^3$ is defined as a particular type of Riemann sum, this time of the function $\mathbf{F} \cdot \mathbf{n}$, but in practice the formula we will use to calculate a surface integral of a vector field is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D (\mathbf{F} \cdot \mathbf{n})(u, v) |\mathbf{r}_u \times \mathbf{r}_v|\, dA,$$

where $\mathbf{r}(u, v)$ is a parameterization of S defined on the region D. We need to calculate $\mathbf{F} \cdot \mathbf{n}$ as a function of u, v using the parameterization $\mathbf{r}$. Recall that if we choose $\mathbf{r}$ such that $\mathbf{r}_u \times \mathbf{r}_v$ points in the same direction as the given orientation $\mathbf{n}$, then

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

Therefore, another way of writing the above formula is as

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D (\mathbf{F} \cdot \mathbf{r}_u \times \mathbf{r}_v)(u, v)\, dA.$$

This formula is analogous to the formula used to calculate the line integral of a vector field:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b (\mathbf{F} \cdot \mathbf{r}')(t)\, dt.$$

In practice, evaluating a surface integral is a lengthy and multi-step calculation. Given a surface S (with a given orientation) and a vector field $\mathbf{F}$, to calculate a surface integral we usually need to do the following:

- Find a parameterization $\mathbf{r}(u, v)$, over a region D in the uv plane, which describes S, and has the correct orientation. In practice, this means that $\mathbf{r}_u \times \mathbf{r}_v$ should be pointing in the same direction as the normal vector $\mathbf{n}$ which describes the given orientation on S. Techniques for finding the parameterization of S depend on the actual surface S you are given; in practice this can be quite hard and is only straightforward for relatively simple surfaces.
- Calculate $\mathbf{r}_u \times \mathbf{r}_v$. This involves taking partial derivatives and computing a cross product. You should check that $\mathbf{r}_u \times \mathbf{r}_v$ points in the same direction as the unit normal vector $\mathbf{n}$ specifying the orientation of S. If you find that $\mathbf{r}_u \times \mathbf{r}_v$ points in the opposite direction of $\mathbf{n}$, you can keep using $\mathbf{r}$ in your calculations, but be sure to reverse the sign of your final answer if you are using an $\mathbf{r}$ of the incorrect orientation.

- In the definition of $\mathbf{F}$, replace all the x, y, zs with the components $X(u, v), Y(u, v), Z(u, v)$ of the parameterization $\mathbf{r}$. Then take the dot product of this with $\mathbf{r}_u \times \mathbf{r}_v(u, v)$ to obtain a scalar function of u, v.
- Evaluate the resulting double integral in u, v over D.

We say that we 'usually' do these steps because there are times when you can simplify the calculation of a surface integral if the problem is of a special form. We will also learn a deep theorem next week which sometimes lets us calculate a surface integral indirectly.

Examples.

- Let S be the sphere $x^2 + y^2 + z^2 = 4$ with the default outward orientation. Let $\mathbf{F}(x, y, z) = \langle x, y, z\rangle$. Calculate the flux of $\mathbf{F}$ across S.

  In this example we will illustrate a shortcut which you can sometimes take in evaluating a surface integral. If we wanted to, we could find a parameterization of the sphere using spherical coordinates, but then all the calculations would be long and messy. We will instead use a special property of this choice of $\mathbf{F}$ to solve this problem with very few computations.

  If you draw a sketch of the vectors $\mathbf{F}$ at points of S, you will find that they are all orthogonal to the surface S, because the geometry of the sphere is such that the normal vectors to a sphere either point radially outwards or radially inwards. In particular, with the specified outward orientation, a point (x, y, z) on S has unit normal vector

  $$\mathbf{n} = \frac{\langle x, y, z\rangle}{|\langle x, y, z\rangle|} = \langle x, y, z\rangle,$$

  since (x, y, z) lies on S so has distance 2 from the origin. Therefore,

  $$\mathbf{F} \cdot \mathbf{n} = \langle x, y, z\rangle \cdot \frac{\langle x, y, z\rangle}{2} = 2,$$

  again because (x, y, z) has distance 2 from the origin. The point of this calculation is that we can easily determine $\mathbf{F} \cdot \mathbf{n}$ at every point of S, and this value is constant, because $\mathbf{F}$ is orthogonal to S at every point of S, so that calculating $\mathbf{F}\cdot\mathbf{n}$ is especially straightforward. Since we know $\mathbf{F} \cdot \mathbf{n} = 2$, the flux of $\mathbf{F}$ across S is

  $$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint\limits_S 2\, dS = 2A(S),$$

  where $A(S)$ is the surface area of S. Since S is a sphere of radius 2, the surface area is $4\pi 2^2 = 16\pi$, so the total flux is equal to 32π.

  Two very special properties had to hold for us to simplify the calculation of a surface integral so much. We needed to be able to compute $\mathbf{F}\cdot\mathbf{n}$ easily by exploiting the geometric relationship of $\mathbf{F}$ with $\mathbf{n}$, and we also needed this value to be constant across S. Even if $\mathbf{F}\cdot\mathbf{n}$ can be easily computed through geometry, if this value varies over S, we would have had to find a parameterization of S to allow us to calculate the surface integral of the scalar function $\mathbf{F} \cdot \mathbf{n}$.
- So far, our definition of surface integrals technically only holds for smooth surfaces S, which have $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ everywhere. We can extend the definition of a surface integral to a piecewise smooth surface S, which is a surface which is smooth everywhere except at a finite number of curves. For example, a polyhedron, such as a tetrahedron or cube, are examples of piecewise smooth surfaces. They are not smooth at their edges (a bit of thought shows that you cannot define a unit normal $\mathbf{n}$ there), but they are smooth at every point except their edges. We can usually

break up such a piecewise smooth surface into its smooth components and then calculate surface integrals for each component separately.

For example, let S be a cube with vertices at (a, b, c), where a, b, c take the values 0 or 1. Let S have the usual outward orientation. If $\mathbf{F} = \langle x + 1, 0, 0\rangle$, find the flux of $\mathbf{F}$ across S.

To solve this problem, we will again note a special geometric relationship between $\mathbf{F}$ and S to simplify the calculation of this surface integral. If we wanted to solve this problem using the general technique outlined above, we would have to break up S into its 6 faces, and then evaluate the flux across each face separately. This would involve finding a parameterization for each face (which is not too hard as they are all pieces of planes, but is still annoying) and then carrying out the usual steps for each face. Obviously, this would be very computationally intensive.

Fortunately, much like the sphere example above, a special geometric relationship between $\mathbf{F}$ and S greatly simplifies our calculations. Consider the four faces which are parallel to the xy or xz planes. The unit normal vectors to these planes are $\pm\mathbf{k}$ or $\pm\mathbf{j}$ respectively, so $\mathbf{F} \cdot \mathbf{n}$ for these faces will always equal 0, because $\mathbf{F}$ has y, z components equal to 0! Therefore, the surface integral of $\mathbf{F}$ across these faces will equal 0.

All that remains are the two faces corresponding to $x = 0, x = 1$, which are parallel to the yz plane. For the face $x = 0$, the unit normal is given by $\langle -1, 0, 0\rangle$, because of the outward orientation of the cube. Therefore, $\mathbf{F} \cdot \mathbf{n} = -1$ on the entire face, since $\mathbf{F} = \langle 1, 0, 0\rangle$ on that face. Therefore, the flux across the face $x = 0$ is equal to -1 times the area of the face, which is 1, so the flux is -1. On the face $x = 1$, we can choose $\mathbf{n} = \langle 1, 0, 0\rangle$, and then $\mathbf{F} \cdot \mathbf{n} = 2$, since $\mathbf{F} = \langle 2, 0, 0\rangle$ on that entire face. This face also has area 1, so the flux across this face is equal to 2. In total, then, the flux across the entire cube is equal to $2 - 1 = 1$.

In this example, we still used the fact that $\mathbf{F} \cdot \mathbf{n}$ could be easily calculated and was constant on each face of the cube to simplify the calculation of a surface integral. On some faces, $\mathbf{F}$ and $\mathbf{n}$ were always orthogonal, while on other faces, they were parallel.

- (Exercise #21, Chapter 17.7 of textbook) Let $\mathbf{F} = \langle xze^y, -xze^y, z\rangle$, and let S be the part of the plane $x + y + z = 1$ in the first octant with downward orientation. Find the flux of $\mathbf{F}$ across S.

  This time we cannot calculate $\mathbf{F} \cdot \mathbf{n}$ as quickly as the previous two examples, but we still are in a situation where our calculations are not as complicated as they otherwise might be. First, we find a parameterization for S; since S is the graph of the function $z = 1 - x - y$ over the domain $x \geq 0, y \geq 0, x + y \leq 1$, we can use

$$\mathbf{r}(u, v) = \langle u, v, 1 - u - v\rangle, u + v \leq 1, 0 \leq u, v.$$

  We calculate $\mathbf{r}_u \times \mathbf{r}_v$ for this choice of $\mathbf{r}$. Since $\mathbf{r}_u = \langle 1, 0, -1\rangle, \mathbf{r}_v = \langle 0, 1, -1\rangle$, $\mathbf{r}_u \times \mathbf{r}_v = \langle 1, 1, 1\rangle$. Notice that this points in the wrong direction, so we should remember to reverse the sign of our answer at the end of our calculations. In any case, for this choice of $\mathbf{r}_u \times \mathbf{r}_v$,

$$\mathbf{F} \cdot \mathbf{r}_u \times \mathbf{r}_v = \langle xze^y, -xze^y, z\rangle \cdot \langle 1, 1, 1\rangle = z.$$

  As a function of u, v, this is equal to $1 - u - v$.

  To find the surface integral we are interested in, we need to calculate the double integral of this function over the domain D of the uv plane which describes S, which is $u, v \geq 0, u + v \leq 1$. This can be described using inequalities $0 \leq u \leq 1, 0 \leq v \leq 1 - u$. Therefore, the integral we want to calculate is equal to

$$\iint_D (1-u-v)\,dA = \int_0^1 \int_0^{1-u} 1-u-v\,dv\,du = \int_0^1 \frac{(1-u)^2}{2}\,du = \frac{-(1-u)^3}{6}\Big|_0^1 = \frac{1}{6}.$$

Recall that we need to reverse the sign, since our choice of $\mathbf{r}$ yielded the incorrect direction of $\mathbf{r}_u \times \mathbf{r}_v$, so the answer is $-1/6$.

In this example, because we could not rapidly calculate $\mathbf{F} \cdot \mathbf{n}$, we carried out all the steps that are usually needed to evaluate a surface integral. Nevertheless, in this example $\mathbf{r}_u \times \mathbf{r}_v$ was constant, so our resulting calculations were relatively simple. Notice that if we choose to just integrate $\mathbf{F} \cdot \mathbf{r}_u \times \mathbf{r}_v$, we can skip the calculation of $\mathbf{n}$.